On Lieb's conjecture *

Horia Scutaru
Department of Theoretical Physics, Institute of Atomic Physics,
PO Box MG-6,
Bucharest-Magurele, Romania,
e-mail: scutaru@theor1.theory.nipne.ro

The reformulation of Lieb's conjecture, in the frame of the harmonuic analysis on the $SO(3)$ group, makes it evident that the exact value of the classical entropy of a pure quantum state, which belongs to the Hilbert space $\mathcal{H}_J$ of a $(2J+1)$-dimensional, unitary, irreducible representation $\mathcal{U}_J$ of the $SO(3)$ group, depends only on the parameters which characterize the orbits of $\mathcal{U}_J$ in $\mathcal{H}_J$. In the case $J = 1$ we give the exact analytic dependence of the classical entropy of a quantum state on the parameters which characterize the orbits and as a consequence we obtain a verification of Lieb's entropy conjecture. We verify this conjecture also for any value of J for the states of the canonical basis of $\mathcal{H}_J$. A natural generalization of Lieb's entropy conjecture, which is a new "phenomenon" in the harmonic analysis on $SO(3)$, is discussed in the case $J = 1$.

I. INTRODUCTION

The present paper is devoted to a verification of Lieb's entropy conjecture[1] in some particular cases. From the begining we point out the connection of this problem with the harmonic analysis on the group $SO(3)$ of rotations in three dimensions. Let $\mathcal{U}_J(g)$, $(g \in SO(3))$ be a unitary irreducible representation of $SO(3)$ in the $(2J+1)$-dimensional Hilbert space $\mathcal{H}_J$, where $J = \frac{1}{2}, 1, \frac{3}{2}, ...,$ and let $\{v_m\}$, $m = -J, -J+1, ..., J-1, J$, be the canonical basis in $\mathcal{H}_J$. We denote by $||.||$ the norm in $\mathcal{H}_J$ and suppose that $||v_m|| = 1$ for any value of m. The matrix elements of the representation $\mathcal{U}_J$ in the canonical basis are denoted by :

$$t^J_{mn}(g) = (v_m, \mathcal{U}_J(g)v_n) = \exp\{-i(m\phi + n\psi)\}t^J_{mn}(\theta) \tag{1,1}$$

where (ϕ, θ, ψ) are the Euler angles which define the rotation $g \in SO(3)$, and

$$t^J_{mn}(\theta) = P^J_{mn}(\theta) \tag{1,2}$$

where

$$P^J_{mn}(\cos\theta) = i^{-m-n}\sqrt{\frac{(J-m)!(J-n)!}{(J+m)!(J+n)!}}(ctg\frac{\theta}{2})^{m+n}$$
$$\sum_{k=max(m,n)}^{J} \frac{(-1)^k(J+k)!}{(J-k)!(k-m)!(k-n)!}(\sin\frac{\theta}{2})^2k \tag{1,3}$$

Because the unitary, irreducible representations of the compact groups are square integrable we have in the particular case of the $SO(3)$ group, for any $u, v \in \mathcal{H}_J$:

$$\frac{2J+1}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |(u, \mathcal{U}_J(g)v)|^2 \sin\theta d\theta d\phi d\psi = ||u||^2||v||^2 \tag{1,4}$$

and from this, for any $u \in \mathcal{H}_J$, we obtain :

$$\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} |(u, \mathcal{U}_J(g)v_{\pm J})|^2 \sin\theta d\theta d\phi = ||u||^2 \tag{1,5}$$

*This paper is the preprint FT-180-1979 which appeared twenty years ago. I have submitted it at that time for publication to the journals Letters in Mathematical Physics and Communications in Mathematical Physics. It was refuted by both these journals and remained unpublished until now. Recently I have discovered the LANL electronic preprint math-ph/9902017 by Peter Schupp entitled "On Lieb's conjecture for the Wehrl entropy of Bloch coherent states" in which some of the results of my paper are rediscovered. This fact determined me to emphasize the existence of my preprint by converting it into an electrronic preprint.

Lieb's conjecture takes in these notations the following form :

$$-\frac{d}{dp}\Big(\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}|(\frac{u}{||u||},\mathcal{U}_J(g)v_{\pm J})|^2\sin\theta d\theta d\phi\Big)|_{p=1}\geq\frac{2J}{2J+1} \tag{1,6}$$

where the equality is attained only for the Bloch coherent states :

$$u=\mathcal{U}_J(g)v_{\pm J} \tag{1,7}$$

for any $g\in SO(3)$. In fact this conjecture may be considered as a consequence of the following conjecture :

$$\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}|(u,\mathcal{U}_J(g)v_{\pm J})|^{2p}\sin\theta d\theta d\phi\leq\frac{2J}{2pJ+1}||u||^{2p} \tag{1,8}$$

where, when $p\geq 1$, the equality is attained only for Bloch coherent states (1,7), and when $p=1$ for any $u\in calH_J$. This last conjecture is in fact a conjecture about the sharp estimation of the $L^{2p}(S^2)$-norms of the matrix coefficients $(u,\mathcal{U}_J(g)v_{\pm J})$:

$$\Big(\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}|(u,\mathcal{U}_J(g)v_{\pm J})|^{2p}\sin\theta d\theta d\phi\Big)^{\frac{1}{2p}}\leq\Big(\frac{2J}{2pJ+1}\Big)^{\frac{1}{2p}}||u|| \tag{1,9}$$

A result of this kind is unknown in the harmonic analysis on the $SO(3)$ group. For the Heisenberg group such a result was proved in[3]. In section 2 we obtain the exact value of the classical entropy of a quantum state[1,4] and as a consequence we verify (1,6) for $J=1$. In section 3 we obtain the exact value of the left hand side of (1,8) and prove (1,6) and (1,8) for any value of J, for the states of the canonical basis: $\{u=v_m; m=-J,-J+1,...,J-1,J\}$. In section 4 we discuss the conjecture (1,8) for $J=1$.

II. THE EXACT VALUE OF THE CLASSICAL ENTROPY OF A QUANTUM STATE FOR J=1

The essential property of the integral:

$$I_p^J(O_u)=\frac{2J+1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}|(u,\mathcal{U}_J(g)v_{\pm J})|^{2p}\sin\theta d\theta d\phi, \tag{2,1}$$

which we shall exploit in the following, is the fact that it depends only on the orbit O_u of $\mathcal{U}_J$ in $\mathcal{H}_J$ to which beelongs the vector $u\in\mathcal{H}_J$. From this property it follows that it is sufficient to calculate the integral $I_p^J(O_u)$ only for one representant from each orbit O_u; this representant may be choosen to be the most simple one. The space $\mathcal{H}_{J=1}$ splits[5] into three strata (union of orbits with the same stabilizer up to conjugacy) which are characterized by a real valued parameter $a\in[0,1]$ which is defined for any vector $u=c_{-1}v_{-1}+c_0v_0+c_1v_1\in\mathcal{H}_1$ in the following way

$$a(u)=\frac{|c_0^2-2c_{-1}c_1|}{|c_{-1}|^2+|c_0|^2+|c_1|^2} \tag{2,2}$$

A typical vector u of a stratum, which is characterized by a given value of this parameter, is of the following form:

$$u=||u||(\sqrt{1-a}v_{-1}+\sqrt{a}v_0) \tag{2,3}$$

The stratum for which $a=0$ contains only the two-dimensional orbit $O_0=O_{v_{-1}}=O_{v_1}$, which is the orbit of Bloch coherent states. The stratum with $a\in(0,1)$ is a continuous set of three-dimensional orbits O_a, one for each value of the parameter a. The stratum for which $a=1$ contains only the two-dimensional orbit $O_1=O_{v_0}$.

We shall obtain the classical entropy[1,4] of a pure quantum state $\frac{u}{||u||}=(\sqrt{1-a}v_{-1}+\sqrt{a}v_0)$, defined by:

$$\mathbf{S}^{cl}(\frac{u}{||u||})=-\frac{d}{dp}I_p^1(O_a)|_{p=1} \tag{2,4}$$

as a function of $a\in[0,1]$. With the notation $x=\cos\theta$ we have:

$$I_p^1(O_a)=\frac{3}{4\pi}\int_{-1}^1dx\int_0^{2\pi}d\phi\left[(1-a)(\frac{1-x}{2})^2+2a(\frac{1-x}{2})(\frac{1+x}{2})+2(2a(1-a)(\frac{1+x}{2})(\frac{1-x}{2})^3)^{\frac{1}{2}}\cos(\phi+\frac{\pi}{2})\right] \tag{2,5}$$

For $a=0$ and $a=1$ we obtain:

$$I_p^1(O_0)=\frac{3}{2p+1} \tag{2,6}$$

and

$$I_p^1(O_1) = \frac{3}{2p+1} \frac{2^p \Gamma(p+1)^2}{\Gamma(2p+1)} \tag{2,7}$$

respectively.

For each $a \in (0,1)$ we split the integral with respect to x in two pieces: one from -1 to $\frac{1-3a}{1+a}$ and other from $\frac{1-3a}{1+a}$ to 1. Further we change the variable in the first integral into $t = \frac{2a(1+x)}{(1-a)(1-x)}$ and in the second integral into $t = \frac{(1-a)(1-x)}{2a(1+x)}$. Then after the integration with respect to ϕ and the use of the formula[6]:

$$F(-p,-p;1;t) = \frac{1}{2\pi} \int_0^{2\pi} (1 + 2t\cos(\phi+\alpha) + t^2)^p d\phi \tag{2,8}$$

we obtain

$$I_p^1(O_a) = \frac{3}{2}\left[\frac{(1-a)^{p+1}}{2a}\int_0^1 dt(\frac{1-a}{2a}t+1)^{-2(p+1)}F(-p,-p;1;t) + \frac{(2a)^{2p+1}}{(1-a)^{p+1}}\int_0^1 dt(1+\frac{2a}{1-a}t)^{-2(p+1)}t^p F(-p,-p;1;t)\right] \tag{2,9}$$

From the fact that the integrands which appear in (2,9) are free of singularities for $t \in [0,1]$, $a \in (0,1)$ and $p \geq 1$, it follows that $I_p^1(O_a)$ is a differentiable function of a and p for $a \in (0,1)$ and $p \geq 1$. We shall calculate the classical entropy (2,4) using this representation for $I_p^1(O_a)$. From the fact that

$$F(-p,-p;1;t) = 1 + p^2 t + (\frac{p(p-1)}{2!})^2 t^2 + (\frac{p(p-1)(p-2)}{3!})^2 t^3 + ... \tag{2,10}$$

and because this series is absolutely converging for all $t \in [0,1]$ we obtain :

$$\frac{d}{dp}F(-p,-p;1;t)|_{p=1} = 2t \tag{2,11}$$

for all $t \in [0,1]$. After tedious calculations we obtain the following simple expression for the classical entropy of a pure quantum state $\frac{u}{||u||} \in O_a$:

$$\mathbf{S}^{cl}(a) = \frac{2}{3} + (a - ln(1+a)) \tag{2,12}$$

for $a \in (0,1)$. When $a=0$ or $a=1$ we obtain directly from (2,6) and (2,7)

$$\mathbf{S}^{cl}(0) = \frac{2}{3} \tag{2,13}$$

and

$$\mathbf{S}^{cl}(1) = \frac{2}{3} + (1 - ln2) \tag{2,14}$$

respectively. We remark that (2,13) and (2,14) are particular cases of (2,12) for $a=0$ and $a=1$ respectively. The relation (2,12) is thus valid for all $a \in [0,1]$. Now Lieb's entropic conjecture for $J=1$:

$$\mathbf{S}^{cl}(1) \geq \frac{2}{3} \tag{2,15}$$

where the equality is attained only for the Bloch coherent states $\frac{u}{||u||} \in O_0$, is a simple consequence of (2,12), (2,13) and of the well known inequality:

$$a - ln(1+a) \geq 0 \tag{2,16}$$

which is valid for all $a \geq 0$. From (2,12) it is obvious that the classical entropy attains its maximum value for $a=1$, i.e., for $\frac{u}{||u||} \in O_{v_0}$.

III. THE VERIFICATION OF LIEB'S CONJECTURE AND OF ITS GENERALIZATION FOR ANY VALUE OF J FOR THE VECTORS OF THE CANONICAL BASIS

We shall calculate the exact value of the integrals $I_p^J(O_{v_m})$ for $m = -J, -J+1, ..., J-1, J$, where $J = \frac{1}{2}, 1, \frac{3}{2}, 2, ...$ and $p \geq 1$. In this case we have

$$I_p^J(O_{v_m}) = \frac{2J+1}{2}\int_{-1}^{1}|P_{m,-J}^J(x)|^{2p}dx = \frac{2J+1}{2}\int_{-1}^{1}|P_{m,J}^J(x)|^{2p}dx \tag{3,1}$$

and obtain

$$I_p^J(O_{v_m}) = \frac{2J+1}{2pJ+1}(\frac{(2J)!}{(J+m)!(J-m)!})^p\frac{\Gamma(p(J-m)+1)\Gamma(p(J+m)+1)}{\Gamma(2pJ+1)} \tag{3,2}$$

From this fornula it is obvious that $I_p^J(O_{v_m}) = I_p^J(O_{v_{-m}})$ for all values of m. The classical entropy of a pure quantum state v_m, $m = -J, -J+1, ..., J-1, J$, is then given by:

$$\mathbf{S}^{cl}(v_m) = (J+m)\left(\frac{1}{J+m+1}+\frac{1}{J+m+2}+...+\frac{1}{2J}\right)+ \\ (J-m)\left(\frac{1}{J-m+1}+\frac{1}{J-m+2}+...+\frac{1}{2J}\right)-\ln(\frac{(2J)!}{(J+m)!(J-m)!})+\frac{2J}{2J+1} \tag{3,3}$$

Since

$$\mathbf{S}^{cl}(v_{-J}) = \mathbf{S}^{cl}(v_J) = \frac{2J}{2J+1} \tag{3,4}$$

it follows that Lieb's entropic conjecture is then equivalent with the following inequality in which we have used the notations $k = J+m$, $j = J-m$:

$$k(\frac{1}{k+1}+\frac{1}{k+2}...+\frac{1}{k+j})+j(\frac{1}{j+1}+\frac{1}{j+2}+...+\frac{1}{j+k}) \geq \ln(\frac{(k+j)!}{k!j!}) \tag{3,5}$$

For $k=1$ we obtain a well known inequality[7]

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{j} \geq \ln(j+1) \tag{3,6}$$

valid for any nonnegative integer j. This inequality is proved by induction, using the well known inequality:

$$\frac{1}{k} \geq \ln(\frac{k+2}{k+1}) \tag{3,7}$$

which is valid for any nonnegative integer k. We can also prove the inequality (3,5) by induction, first with respect to k and finally with respect to j, using (3,7). In this way we have proved Lieb's entropic conjecture for any value of J and for all states v_m, $m = -J, -J+1, ..., J-1, J$. We remark that with the use of inequality (3,7) we may prove that $\mathbf{S}^{cl}(v_m)$ attains it maximum value for $m=0$.

In the following we shall discuss the generalized conjecture:

$$I_p^J(O_{v_m}) \leq \frac{2J+1}{2pJ+1} \tag{3,8}$$

for any value of J and for $p \geq 1$. From (3,2) it follows that this inequality is equivalent with the following inequality for the Γ-function:

$$\frac{\Gamma(kp+1)\Gamma(jp+1)}{\Gamma((k+j)p+1)} \leq \left(\frac{\Gamma(k+1)\Gamma(j+1)}{\Gamma(k+j+1)}\right)^p \tag{3,9}$$

which is unknown. This inequality may be written as an inequality for the B-function:

$$((k+j)p+1)B(kp+1, jp+1) \leq ((k+j+1)B(k+1, j+1))^p \tag{3,10}$$

for any nonnegative integers k and j and any $p \geq 1$. We shall consider the most general inequality:

$$((a+b)p+1)B(ap+1, bp+1) \leq ((a+b+1)B(a+1, b+1))^p \tag{3,11}$$

for any real nonnegative numbers a and b and for any $p \geq 1$. We have a proof of this inequality only for $a = b$. This is based on the integral representation for the B-function[8] (see §1.1,1.6.3) which in the case $a = b$ becomes:

$$\frac{1}{(2b+1)B(b+1,b+1)} = 2^{2b} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{2b} \frac{d\phi}{\pi} \tag{3,12}$$

Then from Jensen's inequality[9] (see chap. 3, Th. 3.3) we have:

$$\begin{gathered} \left(\frac{1}{(2b+1)B(b+1,b+1)}\right)^p = \\ \left(2^{2b} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{2b} \frac{d\phi}{\pi}\right)^p \leq \\ 2^{2pb} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{2pb} \frac{d\phi}{\pi} = \\ \frac{1}{(2bp+1)B(pb+1,pb+1)} \end{gathered} \tag{3,13}$$

Hence, the inequality for $a \neq b$ remains a conjecture.

IV. DISCUSSION OF THE GENERALIZED CONJECTURE IN THE CASE $J = 1$

In this section we shall discuss, for $J = 1$, the conjecture (1,8) which in this case becomes :

$$I_p^1(O_a) \geq \frac{3}{2p+1} \tag{4,1}$$

for any value of $a \in (0,1)$ and for any $p \geq 1$. In order to verify (4,1) we try to find the explicit form of the integral $I_p^1(O_a)$ as a function of the parameter a. First we calculate this integral, in a straightforward manner, in the case in which p is a positive integer ($p = n \geq 1$), and obtain :

$$I_n^1(a) = \frac{3}{2n+1} \sum_{s=0}^{[\frac{n}{2}]} \sum_{r=0}^{n-2s} \sum_{t=0}^{n-s-r} \frac{(-1)^t 2^{s+r}(2n-s-r)!(s+r)!n!a^{s+r+t}}{(s!)^2 r!(n-s-r)!(2n)!(n-s-r-t)1t!} \tag{4,2}$$

After tedious calculations we obtain from this expression that the coefficients of a^{2k+1} are equal to zero for $k = 0, 1$ and that the coefficients of a^{2k} for $k = 1, 2$ are of the following form:

$$I_n^1(a) = \frac{3}{2n+1}\left(1 - \frac{n(n-1)}{2(2n-1)}a^2 + \frac{n(n-1)(n-2)(n-3)}{2^2 2(2n-1)(2n-3)}a^4 - ...\right) \tag{4,3}$$

The comparison of this expression with the following function:

$$\frac{2^n(n!)^2}{(2n)!} a^n P_n(\frac{1}{a}) = \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^k n(n-1)(n-2)...(n-2k+1)}{2^k k!(2n-1)(2n-3)...(2n-2k+1)} a^{2k}, \tag{4,4}$$

where $P_n(\cdot)$ are the Legendre polynomials, suggests that:

$$I_n^1(a) = \frac{3}{2n+1} \frac{2^n(n!)^2}{(2n)!} a^n P_n(\frac{1}{a}). \tag{4,5}$$

If we assume that (4,5) is valid we obtain that:

$$I_n^1(a) \leq \frac{3}{2n+1} \tag{4,6}$$

where the equality is attained only for $a = 0$. Indeed from the fact that thee roots of the Legendre polinomials lie all in the interval $(-1,1)$ and from the fact that if $P_n(b) = 0$ it results that either $b = 0$ or $P_n(-b) = 0$, we obtain that:

$$P_n(x) \leq \frac{(2n)!}{2^n(n!)^2} x^n \tag{4,7}$$

for any $x > 1$. From this inequality we get:

$$\frac{2^n (n!)^2}{(2n)!} a^n P_n(\frac{1}{a}) \leq 1 \tag{4,8}$$

which together with (4,5) gives (4,6). Now we shall try to extend the formula (4,6) to all real values of $p \geq 1$ using spherical functions $P_p(x)$ instead of the Legendre polynomials. Then we shall make the hypothesis that:

$$I_p^1(O_a) = \frac{3}{2p+1} \frac{2^p \Gamma(p+1)^2}{\Gamma(2p+1)} a^p P_p(\frac{1}{a}) \tag{4,9}$$

Then, from the fact that[10] :

$$P_p(z) = (\frac{1+z}{2})^p F(-p,-p;1;\frac{z-1}{z+1}) \tag{4,10}$$

for $Re(z) > 0$, we obtain:

$$I_p^1(O_a) = \frac{3}{2p+1} \frac{2^p \Gamma(p+1)^2}{\Gamma(2p+1)} (\frac{1+a}{2})^p F(-p,-p;1;\frac{1-a}{1+a}) \tag{4,11}$$

From this formula we obtain immediately that:

$$\frac{dI_p^1(O_a)}{dp}|_{p=1} = \frac{2}{3} + (a - \ln(1+a)) \tag{4,12}$$

which coincides with the result proved in section II.

Finally we remark that the inequality (4,1) is equivalent with the following inequality:

$$P_p(x) \leq \frac{\Gamma(2p+1)}{2^p \Gamma(p+1)^2} x^p \tag{4,13}$$

for all $x \geq 1$, or with the inequality:

$$F(-p,-p;1;t) \leq \frac{\Gamma(2p+1)}{2^p \Gamma(p+1)^2} (1+t)^p \tag{4,14}$$

for all $t \in [0,1]$. We do not have a proof for these two last inequalities, which are unknown, for noninteger values of p.

---

[1] Lieb, E. H., Commun. Math. Phys. **62**, 35 (1978).
[2] Kirillov, A.,Éléments de la Théorie des Représentations, Moscow, Edition Mir, 1974.
[3] Scutaru, H., Sharp inequalities for L^p norms of matrix coefficients of square integrable representations of the Heisenberg group and for L^p norms of Wigner functions, Preprint FT-167-79, 1979.
[4] Scutaru, H., Rep. Math. Phys. **15**, 305 (1977).
[5] Daumens, M., and Perroud, M., J. Math. Phys. **18**, 1382 (1977).
[6] Bateman, H., Higher Transcendental Functions, vol. II, McGraw-Hill, New York, 1953.
[7] Mitrinović, D. S., Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[8] Kratzer, A. and Franz, W., Transcendente Functionen, Leipzig, 1960.
[9] Rudin, W., Real and Complex Analysis, McGraw Hill, New York, 1970.
[10] Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Series and Products, Academic Press, New York, 1965.